Covariant derivative of a spinor in a metric-affine space

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1 Introduction

1.1 Why study spinors?

The concepts of scalars and vectors are introduced very early on in the study of physics and maths; most secondary school students are familiar with them. If they move on to undertake a degree in physical sciences, they will encounter tensors too. These objects are fundamental tools in understanding our universe and expressing it mathematically. Spinors, on the other hand, are reserved for advanced level study, despite the fact that they are used to describe the most fundamental objects of the universe.

A person with a physics background first encounters spinors when they start studying relativistic quantum mechanics and in particular deal with the Dirac relativistic electron equation, given by:

$$i\hbar\,\gamma^{\mu}\,\partial_{\mu}\,\Psi(x,t) = mc\,\Psi(x,t) \tag{1}$$

This elegant equation, claimed by many to be among the most beautiful equations of physics, describes the behaviour of fermions, i.e. spin-1/2 particles. Examples of fermionic particles are electrons, protons and neutrons; these particles make up the atoms and hence all matter in the universe. These fundamental particles cannot be described using typical objects like scalars, vectors or tensors; instead, spinors are required. This makes the study of spinors essential and a fundamental aspect of particle physics. These objects are of interest to mathematicians as well because they are associated with the Lorentz group and transformations, thus are useful in differential geometry (further explored in section 6.3).

1.2 An intuitive understanding of spinors

Roughly speaking spinors can be thought of as the *square root of a vector*. They are either two component or four component vector-like objects that transform in a particular way under rotations. In fact, a spinor needs to be rotated by 720° to return to its original position, unlike a vector which obviously requires 'only' 360°. To visualise how a spinor rotates, a helpful analogy is the movement of the dancers' arms in *Pandanggo sa ilaw*, a Filipino candle dance; see [1, min. 1:43]. When the dancer's arm is turned once by 360° it is twisted, but another rotation of the same angle returns the arm to the initial state.

1.3 Aims and objectives

The aim of this text is to consider spinors on a metric-affine space, i.e. a space with torsion and non-metricity, and formulate an expression for the covariant derivative of a spinor on such a space.
To do so, we first introduce the basic notions of tensors and parallel transport.

We then derive an expression for the connection in a metric-affine space. Subsequently, we explore the transformation properties of spinors by reviewing the properties of the Lorentz group and exploring the correspondence between $\mathbb{R}_4$ and the spin space P_1 in which spinors live. Finally, we utilise the concept of parallel transport to define the covariant derivative of a spinor in Minkowski space and then generalise this expression to curved space using the tetrad field.

1.4 Index conventions

In this text we will use lower-case Greek letters $\alpha, \beta, \gamma, \ldots$ to label anholonomic coordinates, i.e. coordinates in flat space. Instead, we will use lower-case Latin letters $i, j, k, \ldots$ for general labelling as well as to label holonomic coordinates, that is, coordinates in curved space.

Upper-case Latin letters from the beginning of the alphabet, $A, B, C, \ldots$, will be reserved for 2-spinors, and thus will range from 1 to 2. Similarly, lower-case Latin letters from the beginning of the alphabet, $a, b, c, \ldots$, will be used for 4-spinors, hence ranging from 1 to 4.

The Einstein summation convention is assumed throughout the text unless otherwise stated.

Finally, we will be working in a natural system of units so that $c = G = \hbar = 1$.

2 Tensor transformations and tensor densities

2.1 Tensors

The general transformation law for a tensor of arbitrary rank (p, q) is:

$$T'^{ij...}{}_{kl...} = \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\cdots\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l}\cdots T^{mn...}{}_{pq...} \tag{2}$$

Note that a tensor of rank (p, q) has p contravariant and q covariant indices. The partial derivatives that appear in the transformation law give the components of the Jacobian matrix that is defined as follows:

$$A^i{}_j = \frac{\partial x'^i}{\partial x^j} \tag{3}$$

The Jacobian matrix plays a role in determining how the volume element in a 4-dimensional spacetime transforms. This transformation is determined by the determinant of the matrix as follows:

$$d^4x' = \left|\frac{\partial x'^i}{\partial x^j}\right| d^4x \tag{4}$$

2.2 Tensor densities

From the transformation of an infinitesimal volume element, we may define a **scalar density** such that its product with the volume element is invariant under coordinate transformations, $\tilde{\mathfrak{s}}'\, d^4x' = \tilde{\mathfrak{s}}\, d^4x$ [3]. Therefore, a scalar density is given by:

$$\tilde{\mathfrak{s}}' = \left|\frac{\partial x^i}{\partial x'^j}\right| \tilde{\mathfrak{s}} \tag{5}$$

The definition can be extended to define a **tensor density** which is a product of a tensor and a scalar density and transforms as:

$$\tilde{\mathfrak{T}}'^{ij...}{}_{kl...} = \left|\frac{\partial x^i}{\partial x'^k}\right| \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\cdots\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l}\tilde{\mathfrak{T}}^{mn...}{}_{pq...} \tag{6}$$

A further extension can be made to define a weight. Densities could be of weight w.

$$\tilde{\mathfrak{T}}'^{ij...}{}_{kl...} = \left|\frac{\partial x^i}{\partial x'^k}\right|^w \frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\cdots\frac{\partial x^p}{\partial x'^k}\frac{\partial x^q}{\partial x'^l}\tilde{\mathfrak{T}}^{mn...}{}_{pq...} \tag{7}$$

Equation (7) is the definition of a **relative tensor**. An absolute tensor or just 'tensor' given by equation (2) is a tensor density/relative tensor of weight 0.

[3] To denote densities we use a Gothic kernel with a tilde on top; Gothic kernels are written in LaTeX using the command `mathfrak`, part of the `amsfonts` package. Note that $\mathfrak{T}$ is a Gothic T.

2.3 Mixing and alternation

Finally, we introduce two processes that will be used throughout this report. The **mixing** over n upper or n lower indices consists in constructing all $n!$ isomers resulting from the permutation of these indices, summing these isomers, and dividing by $n!$. We denote this process with round brackets around the indices. To single out indices, we use the sign $||$. Note that the effect of the round brackets or of the sign $||$ is not stopped by any kind of ordinary brackets. An example of mixing is the following:

$$Q_{(ij|k|l)}{}^{m} = \frac{1}{3!}(Q_{ijkl}{}^{m} + Q_{jlki}{}^{m} + Q_{likj}{}^{m} + Q_{ilkj}{}^{m} + Q_{ljki}{}^{m} + Q_{jikl}{}^{m})$$

The mixing of an object gives the symmetric part of that object.

The **alternation** over n upper or n lower indices is found in the same way as for the mixing with the difference that odd permutation have a negative sign. We denote this process with square brackets around the indices. An example of alternation is the following:

$$Q_{[ij|k|l]}{}^{m} = \frac{1}{3!}(Q_{ijkl}{}^{m} + Q_{jlki}{}^{m} + Q_{likj}{}^{m} - Q_{ilkj}{}^{m} - Q_{ljki}{}^{m} - Q_{jikl}{}^{m})$$

The alternation of an object gives the alternating part of that object.

3 Parallel transport and covariant derivative

Let us consider two points separated infinitesimally in the space-time. We call the first point $P(x^i)$ and the second point $Q(x^i + dx^i)$. Consider a vector field Φ; this field takes the value Φ^k at P and $\Phi^k + d\Phi^k$ at Q.

Now if we consider the difference in the vector field value $d\Phi^k$, we notice that it is not a vector as the vectors at point P and point Q obey different transformation laws. In order to find the difference between the vectors in a meaningful way, they must be brought to the same point. This is achieved by **parallel transport**.

Let us parallel-transport the vector Φ^k at P to Q. It is then given by the vector $\Phi^k + \delta\Phi^k$. This vector can now be subtracted from the original vector at Q:

$$(\Phi^k + d\Phi^k) - (\Phi^k + \delta\Phi^k) = d\Phi^k - \delta\Phi^k \tag{8}$$

The resulting difference is the covariant differential of the vector between the two points:

$$D\Phi^k = d\Phi^k - \delta\Phi^k \tag{9}$$

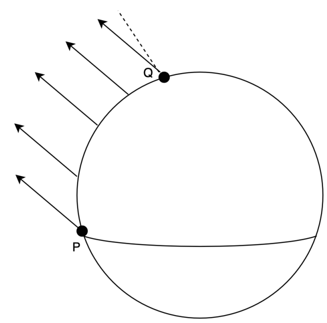

Figure 1: Parallel transport of vector Φ at point P to point Q. The dashed line at Q represents the original vector at Q. The covariant differential is the difference between these two vectors.

In order to define the parallel transport $\delta\Phi$, we introduce the following conditions [8, p. 124]

(i) The covariant differential of a quantity Φ transforms in the same way as Φ.

(ii) The covariant differential of a sum of quantities is the sum of the covariant differentials of the terms.

(iii) The covariant differential obeys the Leibniz rule.

(iv) The covariant differential of a quantity is linear homogeneous in the dx^i.

Therefore, $\delta\Phi$ for contravariant vector can be written as:

$$\delta\Phi^k = -\Gamma^k_{ji}\Phi^j \mathrm{d}x^i \tag{10}$$

where Γ^k_{ji} is an object called *connection*, which will be discussed in section 4. The covariant differential and covariant derivative of a contravariant vector can thus be expressed as:

$$DV^k = dV^k + \Gamma^k_{ji}\, V^j \,\mathrm{d}x^i \tag{11}$$

$$\nabla_i V^k = \partial_i V^k + \Gamma^k_{ji}\, V^j \tag{12}$$

The same expressions for a covariant vector take the following form:

$$DW_k = dW_k - \Gamma^j_{ki}\, W_j \,\mathrm{d}x^i \tag{13}$$

$$\nabla_i W_k = \partial_i W_k - \Gamma^j_{ki}\, W_j \tag{14}$$

The covariant derivative can also act on higher-rank tensors. The general formula is given by [3]

$$\nabla_i\, T^{j\dots}{}_{\ell\dots} = \partial_i\, T^{j\dots}{}_{\ell\dots} + \Gamma^j_{si}\, T^{s\dots}{}_{\ell\dots} + \dots - \Gamma^s_{i\ell}\, T^{j\dots}{}_{s\dots} - \dots \tag{15}$$

4 Affine connection, torsion and non-metricity

4.1 The connection in general relativity

We will now consider the connection Γ^k_{ij}. This is an object that *connects* nearby tangent spaces allowing us to define a covariant derivative on the manifold.

In general relativity, the connection respects the following properties [3]:

i It is *metric-compatible*, i.e. the covariant derivative commutes with the operation of lowering and raising the indices

$$\nabla_i \, g_{jk} = 0 \tag{16}$$

ii It is *torsion-free*, i.e.

$$(\nabla_i \nabla_j - \nabla_j \nabla_i)\, f = 0 \tag{17}$$

which implies that the connection is symmetric on the two lower indices

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{18}$$

Under these conditions, the connection is uniquely defined and coincides with the Christoffel symbol $\{^{\,k}_{ij}\}$ [4], i.e. [3]

$$\Gamma^k_{ij} = \{^{\,k}_{ij}\} = \frac{1}{2} g^{ks} (\partial_i \, g_{js} + \partial_j \, g_{si} - \partial_s \, g_{ij}). \tag{19}$$

However, this is not the most general case possible on a pseudo-Riemannian manifold.

4.2 Torsion

We define a tensor of asymmetry or torsion as the alternating part of the connection, i.e.

$$S_{ij}{}^k = \Gamma^k_{[ij]} \equiv \frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) \tag{20}$$

The connection is said to be

i symmetric if $S_{ij}{}^k = 0$

ii semi-symmetric if $S_{ij}{}^k = S_{[i} A^k_{j]}$

iii asymmetric otherwise.

The torsion tensor can be interpreted in the following way. Consider two linear elements $\underset{1}{d\xi^i}$ and $\underset{2}{d\xi^i}$, and parallelly transport them along each other. In general, the figure so obtained is not a parallelogram, but a pentagon with a closing vector given by $2S_{ij}{}^k \underset{1}{d\xi^i} \underset{2}{d\xi^j}$ [8].

[4] Note that the Christoffel symbol is not a tensor, but instead transforms as

$$\{^{\,k}_{ij}\}' = \frac{\partial X'^k}{\partial X^\ell} \frac{\partial X^m}{\partial X'^i} \frac{\partial X^n}{\partial X'^j} \{^{\,\ell}_{mn}\} + \frac{\partial^2 X^s}{\partial X'^i \partial X'^j} \frac{\partial X'^k}{\partial X^s},$$

since it depends on the partial derivatives of a rank-2 tensor.

4.3 Non-metricity

Consider the metric tensor g_{ij}, which is symmetric and whose inverse is g^{ij}. In general, the covariant derivative of the inverse metric is given by

$$\nabla_i \, g^{jk} = Q_i{}^{jk} \tag{21}$$

which, together with the property $g^{is}g_{sj} = \delta^i_j$, implies

$$\nabla_i \, g_{jk} = -Q_{ijk} \tag{22}$$

where Q_{ijk} is the object of non-metricity [8].
The connection is said to be

i metric if $Q_{ijk} = 0$

ii semi-metric if $Q_{ijk} = Q_i \, g_{ij}$

iii non-metric otherwise.

4.4 General form of the connection

By writing explicitly the covariant derivative of the metric, i.e.

$$\nabla_i \, g_{jk} = \partial_i \, g_{jk} - \Gamma^s_{ij} \, g_{sk} - \Gamma^s_{ik} \, g_{js} = -Q_{ijk}, \tag{23}$$

we find the general form of the connection including torsion and non-metricity [8]:

$$\Gamma^k_{ij} = \{^k_{ij}\} + S_{ij}{}^k - S_j{}^k{}_i + S^k{}_{ij} + \frac{1}{2}(Q_{ij}{}^k + Q_j{}^k{}_i - Q^k{}_{ij}), \tag{24}$$

Equation (24) can be written in a more compact form by using the abbreviation

$$\psi_{\{ijk\}} \equiv \psi_{ijk} - \psi_{jki} + \psi_{kij}. \tag{25}$$

Then, we can define $\chi_{ijk} \equiv \frac{1}{2}\partial_i \, g_{jk}$, so that we can finally write

$$\Gamma^k_{ij} = g^{ks}(\chi_{\{isj\}} - S_{\{isj\}} + \frac{1}{2}Q_{\{isj\}}). \tag{26}$$

5 Lorentz group and transformations

5.1 Definition of the Lorentz group

The Lorentz group is the group of the transformations of the Minkowski space-time, which is the space-time of special relativity expressed as

$$\eta_{\mu\nu}\, x^{\mu}\, x^{\nu} = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2 \tag{27}$$

Note that we are using the convention $(-,-,-,+)$ and that x^4 is the time-like coordinate.

The **homogeneous** Lorentz group is the set of all real linear transformations

$$x^{\mu\prime} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} \tag{28}$$

such that the Minkowski metric is left invariant, and past and future are not interchanged, i.e.

$$\eta_{\mu\nu}\, \Lambda^{\mu}{}_{\alpha}\, \Lambda^{\mu}{}_{\beta} = \eta_{\alpha\beta} \qquad \text{and} \qquad L^4_4 > 0 \tag{29}$$

where $\Lambda^{\mu}{}_{\nu}$ is the **Lorentz matrix** representing a Lorentz transformation [5]. Now taking the determinant of equation (29) gives

$$|\Lambda^{\mu}{}_{\nu}| = \pm 1 \tag{30}$$

- $|\Lambda^{\mu}{}_{\nu}| = +1$ corresponds to **proper Lorentz transformations**.
- $|\Lambda^{\mu}{}_{\nu}| = -1$ corresponds to **improper Lorentz transformations**.

In our investigation, we only focus on the proper Lorentz transformations, which form an invariant subgroup.

5.2 Infinitesimal Lorentz transformations

Let's consider infinitesimal Lorentz transformations which differ from the identity transformation by infinitesimals $\omega_{\mu\nu}$, so the Lorentz matrix is expressed as:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{31}$$

The orders higher than the linear order of ω are negligible.

In any representation of the Lorentz group the infinitesimal transformation equation (31) is represented by:

$$1 + \frac{1}{2}\omega^{\mu\nu}\, G_{\mu\nu} \tag{32}$$

where $G_{\mu\nu}$ are the **generators** of the Lorentz group, 3 of which are given in table 1 in section 6.

6 Spinors and spin space

We can now finally introduce the objects of study, i.e. spinors. We will start by defining them and stating their particular properties. Then, we will discuss a useful correspondence between $\mathbb{R}_4$ and the spin space, which is the space where spinors live. Once this is done, we will be able to write down the infinitesimal transformation of a 2-spinor. Finally, we will extend the theory to 4-spinors and express their infinitesimal transformation as well. These transformations will be needed to define the covariant derivative of a spinor, which is our objective.

Note that we will be working in Minkowski space with metric $\eta_{\mu\nu}$. This is because it is much simpler to work in flat space and then generalise our results to curved space using the tetrads, which will be discussed in section 7.

6.1 Definition of 2-spinors

A *spinor* is a geometrical object ψ_A that is defined over a 2D complex space P_1, called the *spin space*, and that transforms in the following way:

$$\psi'_A = t^B{}_A\,\psi_B \qquad (A, B = 1, 2) \tag{33}$$

where $t^B{}_A$ is in general complex and non-singular.

For our purposes, i.e. physical applications of spinor theory in special relativity, we can restrict ourselves to unimodular matrices $s^A{}_B$, so that [5]

$$\psi'_A = s^B{}_A\,\psi_B. \tag{34}$$

A quantity that transforms in this way is called a *covariant regular spinor of first rank*.

Analogously, a geometric object ψ^A that transforms according to

$$\psi^{A'} = S^A{}_B\,\psi^B \tag{35}$$

where

$$S^A{}_B\,s^B{}_C = \delta^A_C \tag{36}$$

is called a *contravariant regular spinor of first rank.* [5]

Covariant and contravariant components are related using the rules

$$\psi_A = \epsilon_{AB}\,\psi^B \tag{37}$$

$$\psi^A = \epsilon^{BA}\,\psi_B \tag{38}$$

where the metric spinors ϵ_{AB} and ϵ^{AB} are given by

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{AB}. \tag{39}$$

Note that $\epsilon^{BA}\epsilon_{BC} = \delta^A_C$. This is the only way we have to raise and lower indices in this space. [5]

If a quantity is written with one or more dotted indices [5], e.g. $\chi^B{}_{\dot{A}}$, then the corresponding transformation matrix is complex conjugated. For instance,

$$\chi'^B{}_{\dot{A}} = \bar{s}^C{}_A \, S^B{}_D \, \chi^D{}_{\dot{C}}.$$

To raise or lower dotted indices, the metric spinors $\epsilon^{\dot{A}\dot{B}}$ and $\epsilon_{\dot{A}\dot{B}}$ have to be used in the rules

$$\psi_{\dot{A}} = \epsilon_{\dot{A}\dot{B}} \, \psi^{\dot{B}} \tag{40}$$

$$\psi^{\dot{A}} = \epsilon^{\dot{B}\dot{A}} \, \psi_{\dot{B}} \tag{41}$$

where

$$\epsilon_{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon^{\dot{A}\dot{B}}. \tag{42}$$

6.2 Spin-tensor

A quantity with both tensor and spinor indices is called a *tensor-spinor* or a *spin-tensor*. Consider, for instance, the object $\phi_{\mu\dot{A}B}$. It transforms as

$$\phi'_{\mu\dot{A}B} = a^\nu{}_\mu \, \phi_{\nu\dot{A}B}$$

under coordinate transformations in $\mathbb{R}_4$, and as

$$\phi'_{\mu\dot{A}B} = \bar{s}^C{}_A \, s^D{}_B \, \phi_{\mu\dot{C}D}$$

under constant spin transformations in P_1.

The corresponding contravariant transformations are

$$\phi'^{\mu\dot{A}B} = A^\mu{}_\nu \, \phi^{\nu\dot{A}B}$$

$$\phi'^{\mu\dot{A}B} = \bar{S}^A_C \, S^B_D \, \phi^{\mu\dot{C}D}$$

Under combined (constant) coordinate transformations in $\mathbb{R}_4$ and P_1, the spin-tensor $\phi_{\mu\dot{A}B}$ transforms according to

$$\phi''_{\mu\dot{A}B} = a^\nu{}_\mu \, \phi'_{\nu\dot{A}B} = a^\nu{}_\mu \, \bar{s}^C{}_A \, s^D{}_B \, \phi_{\nu\dot{C}D}$$

$$\phi''^{\mu\dot{A}B} = A^\mu{}_\nu \, \phi'^{\nu\dot{A}B} = A^\mu{}_\nu \, \bar{S}^A_C \, S^B_D \, \phi^{\nu\dot{C}D}$$

[5] We will call spinors with dotted indices *conjugated* or *dotted*.

6.3 Correspondence between $\mathbb{R}_3$ and spin space

We now approach the relation between 2-spinors and 4-tensors. In fact, there exists a $(1-1)$ correspondence between Hermitian matrices of second order and points of $\mathbb{R}_4$, thus every second-rank Hermitian matrix can be written in terms of four real parameters p, q, r, s, as, for instance, [5]

$$X_{\dot{A}B} = \begin{pmatrix} p+q & r+is \\ r-is & p-q \end{pmatrix}. \tag{43}$$

Given a 4-vector x^i, we use a formalism in which the identification chosen is

$$p = x^4,\ q = x^3,\ r = x^1,\ s = x^2 \tag{44}$$

so that equation (43) takes the form

$$X_{\dot{A}B} = \begin{pmatrix} x^4+x^3 & x^1+ix^2 \\ x^1-ix^2 & x^4-x^3 \end{pmatrix}. \tag{45}$$

We will now show how this correspondence can be used to find the representation of spatial rotations in spin space.

6.3.1 Rotation about the z axis

For simplicity, consider a general 3D column vector $\mathbf{v} = (v_1, v_2, v_3)^T$ rotated counterclockwise about the z axis by an angle θ. Then, the new rotated vector is given by $\mathbf{v}' = \mathbf{R}\,\mathbf{v}$, i.e.

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \tag{46}$$

$$= \begin{pmatrix} \cos\theta\, v_1 - \sin\theta\, v_2 \\ \sin\theta\, v_1 + \cos\theta\, v_2 \\ v_3 \end{pmatrix} \tag{47}$$

Using equation (45), we can write the 3D vector $\mathbf{v}$ (setting $p = x^4 = 0$) as

$$V_{\dot{A}B} = \begin{pmatrix} v_3 & v_1+iv_2 \\ v_1-iv_2 & -v_3 \end{pmatrix} \tag{48}$$

and similarly the 3D $\mathbf{v}'$ as

$$V'^{(z)}_{\dot{A}B} = \begin{pmatrix} v_3 & \cos\theta\, v_1 - \sin\theta\, v_2 + i(\sin\theta\, v_1 + \cos\theta\, v_2) \\ \cos\theta\, v_1 - \sin\theta\, v_2 - i(\sin\theta\, v_1 + \cos\theta\, v_2) & -v_3 \end{pmatrix} \tag{49}$$

$$= \begin{pmatrix} v_3 & (\cos\theta + i\sin\theta)\, v_1 + (-\sin\theta + i\cos\theta)\, v_2 \\ (\cos\theta - i\sin\theta)\, v_1 - (\sin\theta + i\cos\theta)\, v_2 & -v_3 \end{pmatrix} \tag{50}$$

$$= \begin{pmatrix} v_3 & e^{i\theta}\,(v_1+iv_2) \\ e^{-i\theta}\,(v_1-iv_2) & -v_3 \end{pmatrix} \tag{51}$$

where in the last passage we have used the following relations:

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{52}$$

$$\cos\theta - i\sin\theta = e^{-i\theta} \tag{53}$$

$$\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta) = ie^{-i\theta} \tag{54}$$

$$-\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta} \tag{55}$$

The second-rank Hermitian matrix corresponding to the vector $\mathbf{v}'$, i.e. $V'_{\dot{A}B}$, can be written as [5]

$$\mathbf{V}'^{(z)} = \mathbf{S}^{(z)}\,\mathbf{V}\,\mathbf{S}^{-1\,(z)}. \tag{56}$$

The change of basis matrix $\mathbf{S}$ can be found to be

$$\mathbf{S}^{(z)} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \tag{57}$$

with $\mathbf{S}^{-1} = \mathbf{S}^{\dagger}$.

This is a 2D rotation by the angle $\theta/2$, which is half of the angle by which we had rotated the vector $\mathbf{v}$ in $\mathbb{R}_3$.

Finally, Taylor expanding to first order, we obtain

$$\mathbf{S}^{(z)} \approx \begin{pmatrix} 1 + i\theta/2 & 0 \\ 0 & 1 - i\theta/2 \end{pmatrix} = \mathbf{I} + i\frac{\theta}{2}\,\boldsymbol{\sigma}_z \tag{58}$$

where $\mathbf{I}$ is the 2x2 identity matrix and $\boldsymbol{\sigma}_z$ is the Pauli matrix

$$\boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{59}$$

Now consider the 3D rotation matrix in equation (46). Taylor expanding to first order, we find

$$\mathbf{R} \approx \begin{pmatrix} 1 & -\theta & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} - i\theta \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{I} - i\theta \begin{pmatrix} \boldsymbol{\sigma}_y & 0 \\ 0 & 0 \end{pmatrix} \tag{60}$$

where $\mathbf{I}$ is here the 3x3 identity matrix and $\boldsymbol{\sigma}_y$ is the Pauli matrix

$$\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{61}$$

6.3.2 Rotation about the x axis

Now, let's consider the rotation of the general vector $\mathbf{v} = (v_1, v_2, v_3)^T$ by a small angle θ about the x axis. This gives

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \tag{62}$$

$$= \begin{pmatrix} v_1 \\ \cos\theta\, v_2 - \sin\theta\, v_3 \\ \sin\theta\, v_2 + \cos\theta\, v_3 \end{pmatrix} \tag{63}$$

The corresponding Hermitian matrix representation of equation (62) is

$$V'^{(x)}_{\dot{A}B} = \begin{pmatrix} \sin\theta\ v_2 + \cos\theta\ v_3 & v_1 + i(\cos\theta\ v_2 - \sin\theta\ v_3) \\ v_1 - i(\cos\theta\ v_2 - \sin\theta\ v_3) & -\sin\theta\ v_2 - \cos\theta\ v_3 \end{pmatrix} \tag{64}$$

It can be shown that the change of basis matrix for the similarity transform, $\mathbf{V}'^{(x)} = \mathbf{S}^{(x)}\, \mathbf{V}\, \mathbf{S}^{-1\,(x)}$, is:

$$\mathbf{S}^{(x)} = \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{65}$$

where the following double-angle identities were used:

$$\sin\theta = 2\sin\theta/2\cos\theta/2 \tag{66}$$

$$\cos\theta = \cos^2\theta/2 - \sin^2\theta/2 \tag{67}$$

Taylor expanding $\mathbf{S}^{(x)}$ to first order we obtain

$$\mathbf{S}^{(x)} \approx \begin{pmatrix} 1 & i\theta/2 \\ i\theta/2 & 1 \end{pmatrix} = \mathbf{I} + i\frac{\theta}{2}\boldsymbol{\sigma}_x \tag{68}$$

where $\mathbf{I}$ is here the 3x3 identity matrix and $\boldsymbol{\sigma}_x$ is the Pauli matrix

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{69}$$

6.3.3 Rotation about the y axis

Rotating the general vector $\mathbf{v} = (v_1, v_2, v_3)^T$ about the y-axis gives

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \tag{70}$$

$$= \begin{pmatrix} \cos\theta\, v_1 + \sin\theta\, v_3 \\ v_2 \\ -\sin\theta\, v_1 + \cos\theta\, v_3 \end{pmatrix} \tag{71}$$

The corresponding Hermitian matrix representation of equation (70) is

$$V'^{(y)}_{\dot{A}B} = \begin{pmatrix} -\sin\theta\, v_1 + \cos\theta\, v_3 & \cos\theta\, v_1 + \sin\theta\, v_3 + iv_2 \\ \cos\theta\, v_1 + \sin\theta\, v_3 - iv_2 & \sin\theta\, v_1 - \cos\theta\, v_3 \end{pmatrix} \tag{72}$$

The corresponding similarity transformation for $\mathbf{V}'^{(y)}$ is given by

$$\mathbf{S}^{(y)} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{73}$$

Taylor expanding $\mathbf{S}$ to first order we obtain

$$\mathbf{S}^{(y)} \approx \begin{pmatrix} 1 & -\theta/2 \\ \theta/2 & 1 \end{pmatrix} = \mathbf{I} + i\frac{\theta}{2}\boldsymbol{\sigma}_y \tag{74}$$

where $\mathbf{I}$ is here the 3x3 identity matrix and $\boldsymbol{\sigma}_y$ is the Pauli matrix

$$\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{75}$$

As a result, the Pauli matrices

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{76}$$

are shown to be the generators of infinitesimal rotations in spin space.

The extension to $\mathbb{R}_4$ is trivial when only rotations are considered. [2]

	$\mathbb{R}_4(x,y,z,t)$	Spin-space	Generators
R_x	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$	$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
R_y	$\begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$	$\boldsymbol{\sigma}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
R_z	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \exp(i\theta/2) & 0 \\ 0 & \exp(-i\theta/2) \end{pmatrix}$	$\boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Table 1: Generators of the Lorentz group and the correspondence between spin space and $\mathbb{R}_4$

6.4 Remark on spinor rotations

Spinors appear to be similar to vectors, but differ from them by their behaviour under rotations. Indeed, a vector is left unchanged when rotated by 360°. However, a spinor is not; instead, we obtain its opposite. A rotation of 720° is required to return to the original position. This is because, as we have seen, a spatial rotation by an angle θ is mapped to a rotation in spin space by half that angle.

6.5 The g-spin tensors

Consider again equation (45) and note it can be written as

$$X_{\dot{A}B} = g_{\mu\dot{A}B}\, x^{\mu} \tag{77}$$

where $g_\mu = (\boldsymbol{\sigma_x}, \overline{\boldsymbol{\sigma_y}}, \boldsymbol{\sigma_z}, \mathbf{I})$ and the bar represents complex conjugation.

The inverse equations can be written as

$$x^{\mu} = \frac{1}{2}\, g^{\mu\dot{A}B}\, X_{\dot{A}B} \tag{78}$$

where $g^\mu = (\boldsymbol{\sigma_x}, \boldsymbol{\sigma_y}, \boldsymbol{\sigma_z}, \mathbf{I})^T$.

Note that we have used $g_\mu = \left\| g_{\mu\dot{A}B} \right\|$ and $g^\mu = \left\| g^{\mu\dot{A}B} \right\|$.

Note also that tensor and spinor indices are raised or lowered using the metric tensor and the (skew) metric spinors, respectively.
For a more complete analysis of g-spin tensors, see [5].

6.6 Infinitesimal transformations of 2-spinors

Let us consider Lorentz and spinor transformations; to every proper Lorentz transformation there exists a spin transformation such that the application of two together leaves the spin-tensors ${g^{\mu}}_{\dot{A}B}$ invariant. [5]

Lorentz	Spinor
$x'_\mu = x_\mu + {\omega_\mu}^\nu x_\nu$	$\phi'_A = \phi_A + \eta_A^B \phi_B$
	$\phi'_{\dot{A}} = \phi_{\dot{A}} + \eta_{\dot{A}}^{\dot{B}} \phi_{\dot{B}}$

Table 2: Infinitesimal Lorentz and spinor transformations.

By the above statement, it should be noted that for every $\omega^{\mu\nu}$ there exists η_{AB} such that

$$\omega^{\mu\nu} g_{\nu\dot{C}A} + {\eta^B}_A {g^\mu}_{\dot{C}B} + {\eta^{\dot{D}}}_{\dot{C}} {g^\mu}_{\dot{D}A} = 0 \tag{79}$$

The solution is

$$\eta_{AB} = \frac{1}{4} g_{\mu\dot{C}A} {g_\nu}^{\dot{C}}{}_B \tag{80}$$

The transformation matrices are thus

$$\begin{aligned} &\delta^B{}_A + \frac{1}{4}\omega^{\mu\nu} g_{\mu\dot{C}A} g_\nu{}^{\dot{C}B} \\ &\delta^{\dot{B}}{}_{\dot{A}} + \frac{1}{4}\omega^{\mu\nu} g_{\mu\dot{A}C} g_\nu{}^{\dot{B}C} \end{aligned} \quad (81)$$

By analogy with the representation of the Lorentz group,

$$||1 + \frac{1}{2}\omega^{\mu\nu} G_{\mu\nu}|| = \delta^B{}_A + \frac{1}{4}\omega^{\mu\nu} g_{\mu\dot{C}A} g_\nu{}^{\dot{C}B} \quad (82)$$

we find that,

$$||G_{\mu\nu}|| = \frac{1}{4}(g_{\mu\dot{C}A} g_\nu{}^{\dot{C}B} - g_{\nu\dot{C}A} g_\mu{}^{\dot{C}B}) \quad (83)$$

Now let us consider the infinitesimal transformation of a 2-spinor ϕ_A.

$$\delta\phi_A = \eta_A^B \phi_B \quad (84)$$

$$\delta\phi_A = -\frac{1}{4}\omega^{\mu\nu}\left\{\frac{1}{2}(g_\mu{}^{\dot{C}}{}_A g_{\nu\dot{C}}{}^B - g_\nu{}^{\dot{C}}{}_A g_{\mu\dot{C}}{}^B)\right\}\phi_B \quad (85)$$

We could express this equation in compact form as

$$\delta\phi = \frac{1}{2}\omega^{\mu\nu} G_{\mu\nu \text{ op}}\, \phi \quad (86)$$

6.7 Definition of 4-spinors

4-spinors are a combination of two simple 2-spinors and can be expressed in terms of one covariant regular spinor and one covariant conjugated spinor. [5]

$$\Psi^a = \begin{Vmatrix} \Psi^1 \\ \Psi^2 \\ \Psi^3 \\ \Psi^4 \end{Vmatrix} = \begin{Vmatrix} \psi_1 \\ \psi_2 \\ \phi^{\dot{1}} \\ \phi^{\dot{2}} \end{Vmatrix} \quad (87)$$

6.8 Gamma Matrices

The gamma matrices can be expressed in terms of g-spin tensor components (which are 2×2 matrices) as: [5]

$$\gamma^{\mu a}{}_b = \begin{Vmatrix} 0 & -i\left\|\bar{g}^\mu{}_{\dot{A}B}\right\| \\ i\left\|\bar{g}^{\mu\dot{A}B}\right\| & 0 \end{Vmatrix} \qquad \mu = \{1,2,3,4\} \quad (88)$$

We could also introduce a fifth gamma matrix that satisfies the following relation:

$$\gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0 \quad (89)$$

This gamma matrix can be expressed as:

$$\gamma_5 = \begin{Vmatrix} I & 0 \\ 0 & -I \end{Vmatrix} \tag{90}$$

All gamma matrices are explicitly stated below:

$$\gamma^{1a}{}_b = \begin{Vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{Vmatrix} \quad \gamma^{2a}{}_b = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{Vmatrix}$$

$$\gamma^{3a}{}_b = \begin{Vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{Vmatrix} \quad \gamma^{4a}{}_b = \begin{Vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{Vmatrix}$$

$$\gamma^{5a}{}_b = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}$$

In flat space, the gamma matrices obey the following anti-commutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{91}$$

In curved space, they obey:

$$\{\gamma^i, \gamma^j\} = \gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij} \tag{92}$$

which are interchangeable via the application of tetrads (see section **??**):

$$g_{ij} = e_i{}^\mu e_j{}^\nu \eta_{\mu\nu}. \tag{93}$$

6.9 Infinitesimal transformation of 4-spinors

In a similar manner to 2-spinors, we consider the infinitesimal transformation of 4-spinors [6] [5]

$$\delta\Psi^\bullet = \begin{Vmatrix} \delta\psi_B \\ \delta\phi^{\dot{B}} \end{Vmatrix} = \begin{Vmatrix} \eta^A{}_B\psi_A \\ -\eta^{\dot{B}}_{\dot{A}}\phi^{\dot{A}} \end{Vmatrix} = \frac{1}{4}\omega^{\mu\nu} \begin{Vmatrix} g_{\mu\dot{C}}{}^A g_\nu{}^{\dot{C}}{}_B \psi_A \\ -\bar{g}_{\mu\dot{C}A}\bar{g}_\nu{}^{\dot{C}B}\phi^{\dot{A}} \end{Vmatrix} \tag{94}$$

This can be written in matrix form as:

$$\frac{1}{4}\omega^{\mu\nu} \begin{Vmatrix} g_{\mu\dot{C}}{}^A g_\nu{}^{\dot{C}}{}_B & 0 \\ 0 & -\bar{g}_{\mu\dot{C}A}\bar{g}_\nu{}^{\dot{C}B} \end{Vmatrix} \cdot \begin{Vmatrix} \psi_A \\ \phi^{\dot{A}} \end{Vmatrix} \tag{95}$$

By the anti-symmetry of two indices of $\omega^{\mu\nu}$ this is written as:

[6] The bullets correspond to any lower-case Latin (spinor) index.

$$\frac{1}{4}\omega^{\mu\nu}\left\|\begin{matrix} \frac{1}{2}(g_{\mu\dot{C}}{}^{A}g_{\nu}{}^{\dot{C}}{}_{B} - g_{\nu\dot{C}}{}^{A}g_{\mu}{}^{\dot{C}}{}_{B}) & 0 \\ 0 & \frac{1}{2}(\bar{g}_{\nu\dot{C}A}\bar{g}_{\mu}{}^{\dot{C}B} - \bar{g}_{\mu\dot{C}A}\bar{g}_{\nu}{}^{\dot{C}B}) \end{matrix}\right\|\Psi^{\bullet} \tag{96}$$

In compact form we have that the infinitesimal transformation of a 4-spinor is given by:

$$\delta\Psi^{\bullet} = \frac{1}{2}\omega^{\mu\nu}\, G_{\mu\nu}{}^{\bullet}{}_{\bullet}\, \Psi^{\bullet} \tag{97}$$

where $\omega_{\mu\nu}$ is the infinitesimal rotation and $G_{\mu\nu}{}^{\bullet}{}_{\bullet}$ is the generator of the Lorentz group, i.e.

$$G_{\mu\nu} = \frac{1}{2}\gamma_{\mu\nu} = \frac{1}{4}[\gamma_{\mu}, \gamma_{\nu}]. \tag{98}$$

7 Tetrads

In this section we will show how the results obtained in Minkowski space in the previous sections can be extended to a general curved space using the tetrad frame.

This formalism corresponds to picturing spacetime as populated with observers who can measure spatial and temporal distances, and relative orientations.

In fact, an orthonormal frame of four vectors is introduced at every point in spacetime [6]

$$\mathbf{e}_\alpha(x^k) = e^i{}_\alpha\,\partial_i \qquad \text{with} \qquad \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \eta_{\alpha\beta} = diag(-1,-1,-1,+1). \tag{99}$$

This tetrad field $\mathbf{e}_\alpha(x^k)$ represents the local observers.

The dual frame (co-frame) is given by [7]

$$\mathbf{e}^\alpha = e_i{}^\alpha\,\mathrm{d}x^i \tag{100}$$

so that we have $e_i{}^\alpha\, e^i{}_\beta = \delta^\alpha_\beta$. Similarly, $e_i{}^\alpha\, e^j{}_\alpha = \delta^j_i$.

Any vector can be expressed either using its components v^i with respect to the holonomic coordinate system, or by its coordinate-invariant projections v^α onto the tetrad field $e_i{}^\alpha$ [7]

$$v^\alpha = e_i{}^\alpha\, v^i \tag{101}$$

$$v^i = e^i{}_\alpha\, v^\alpha \tag{102}$$

The metric tensor of the manifold can be related to the local Minkowski metric by using the tetrad

$$g_{ij} = e_i{}^\alpha\, e_j{}^\beta\, \eta_{\alpha\beta}. \tag{103}$$

Analogously, the Minkowski metric can be expressed as

$$\eta_{\alpha\beta} = e^i{}_\alpha\, e^j{}_\beta\, g_{ij}, \tag{104}$$

thus the $e^i{}_\alpha$ matrices are similarity transformations that diagonalise the metric g_{ij} locally to the Minkowski metric.

The tetrad has 16 independent components which determine the 10 components of the metric tensor. The remaining 6 degrees of freedom do not affect the metric, they correspond to the 6 Lorentz transformations of the tangent space.

[7] Recall that a vector can be written using its basis $\mathbf{e}_i$ or its dual basis $\mathbf{e}^i$ as $\mathbf{v} = v^i\,\mathbf{e}_i = v_i\,\mathbf{e}^i$ where v^i and v_i are respectively the *contravariant* and *covariant* components of the vector. The dual basis is such that $\mathbf{e}_i \cdot \mathbf{e}^j = \delta^j_i$, and we can identify $\mathbf{e}_i \leftrightarrow \partial_i$ and $\mathbf{e}^i \leftrightarrow \mathrm{d}x^i$.

Note that holonomic indices are lowered and raised using the metric g_{ij} and its inverse g^{ij}, while anholonomic (coordinate-independent) indices are lowered and raised using the Minkowski metric $\eta_{\alpha\beta}$ and its inverse $\eta^{\alpha\beta}$.

To change from a tetrad to another, we express the vectors of the new tetrad as linear combinations of the vectors of the old one, i.e.

$$\tilde{e}^i{}_\alpha = \Lambda^\beta{}_\alpha \, e^i{}_\beta \tag{105}$$

Applying equation (103) on the new tetrad $\tilde{e}^i{}_\alpha$ gives the orthogonality condition

$$\Lambda^\gamma{}_\alpha \, \Lambda^\delta{}_\beta \, \eta_{\gamma\delta} = \eta_{\alpha\beta} \tag{106}$$

which has to be obeyed by the matrix Λ. This corresponds to the condition given in equation (29), thus Λ is a Lorentz matrix.

Hence, the Lorentz group can be thought as the group of *tetrad rotations* in general relativity [7].

8 The covariant derivative of a spinor and the Dirac equation

8.1 Covariant derivative of a spinor

We will now use the results obtained in the previous sections to express the covariant derivative of a spinor in a metric-affine space. We will first consider flat space, then extend the validity of the equation to a general curved space using the tetrad field.
Finally, we use our result to express the Dirac equation in a general metric-affine space.

We have seen that the infinitesimal transformation of a 4-spinor is given by

$$\delta\Psi = \frac{1}{2}\omega_{\mu\nu}\, G^{\mu\nu}\,\Psi \tag{107}$$

where $G^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ and $\omega_{\mu\nu}$ is an infinitesimal rotation.

Writing $\omega_{\mu\nu} = \omega_{\alpha\mu\nu}\,\mathrm{d}x^\alpha$, we obtain

$$\delta\Psi = \frac{1}{2}\omega_{\alpha\mu\nu}\, G^{\mu\nu}\,\Psi\,\mathrm{d}x^\alpha = \delta\Psi_\alpha\,\mathrm{d}x^\alpha. \tag{108}$$

Then, the covariant differential is given by

$$D\Psi = d\Psi + \delta\Psi_\alpha\,\mathrm{d}x^\alpha \tag{109}$$

and hence the covariant derivative is [8]

$$\nabla_\alpha\Psi = \partial_\alpha\Psi + \delta\Psi_\alpha = \partial_\alpha\Psi + \frac{1}{2}\omega_{\alpha\mu\nu}\, G^{\mu\nu}\,\Psi. \tag{110}$$

Using the tetrad field, we can now express the covariant derivative in curved space:

$$\nabla_i\Psi = e_i{}^\alpha\,\nabla_\alpha\Psi = e_i{}^\alpha(\partial_\alpha\Psi + \frac{1}{2}\omega_{\alpha\mu\nu}\, G^{\mu\nu}\,\Psi) \tag{111}$$

$$\nabla_i\Psi = \partial_i\Psi + \frac{1}{2}\omega_{i\mu\nu}\, G^{\mu\nu}\,\Psi. \tag{112}$$

We identify the object $\omega_{i\mu\nu}$ as the *affine spin connection*. This is defined as [4]

$$\omega_i{}^\alpha{}_\beta = e_k{}^\alpha\,\Gamma^k_{ij}\,e^j{}_\beta + e_j{}^\alpha\,\partial_i\,e^j{}_\beta, \tag{113}$$

where Γ^k_{ij} is the metric-affine connection given by equation (26).

We can lower anholonomic indices using the Minkowski metric, thus

$$\omega_{i\mu\nu} = \eta_{\mu\gamma}\,\omega_i{}^\gamma{}_\nu = \eta_{\mu\gamma}\,(e_k{}^\gamma\,\Gamma^k_{is}\,e^s{}_\nu + e_s{}^\gamma\,\partial_i\,e^s{}_\nu). \tag{114}$$

[8] Cfr. section 3.

8.2 The Dirac equation

Finally, substituting the partial derivative with the covariant derivative, we find that the Dirac equation in a metric-affine space takes the form

$$i\,\gamma^i\,\nabla_i\,\Psi = m\,\Psi \tag{115}$$

where $\nabla_i\,\Psi$ is given by equation (112).

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